A Note Concerning the Utility of

Factored Generating Functions

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Introduction. Alexander Shpilkin[1] has discovered on the web a derivation of the differential equations satisfied by the Chebyshev polynomials $T_n(x)$ that is—by any measure, but especially when compared with that obtained by my general recursive method[2]—remarkable for its swift efficiency. Shpilkin's argument proceeds, however, from a description of $T_n(x)$ that cannot be found in any of the standard sources (Abramowitz & Stegun, Spanier & Oldham). My primary objective here will be to trace Shpilkin's argument to its more familiar roots. That will be accomplished by appeal to the resources latent in a factorization scheme (described below) borrowed from Ray Mayer's approach to a similar problem (derivation of the the differential equations satisfied by certain "Sebbar polynomials). I will look then to light that the scheme can shed on properties of some other polynomials, and to some related matters.

The factorization scheme. Eleven of the generators of orthogonal polynomials listed in Abramowitz & Stegun's Table 22.9 contain the construction

$$R = \sqrt{1 - 2xh + h^2}$$

The scheme proceeds from the observation that

$$1 - 2xh + h^2 = (\alpha - h)(\beta - h) \tag{1}$$

entails

$$\begin{aligned} \alpha\beta &= 1 \\ \alpha + \beta &= 2x \end{aligned} \tag{2}$$

of which the solution (unique to within permutation) is

[1] Private communication, 29 December 2017.

[2] "Extracting differential equations from the generators of polynomials," (November 2017, pages 5–7.

$$\begin{aligned}\alpha(x) &= x + \sqrt{x^2 - 1}\\ \beta(x) &= x - \sqrt{x^2 - 1}\end{aligned} \tag{3}$$

Dividion of (1) by $\alpha\beta = 1$ gives

$$1 - 2xh + h^2 = (1 - h/\alpha)(1 - h/\beta) \tag{4.1}$$
$$= (1 - h/\alpha)(1 - \alpha h) \tag{4.2}$$

Chebyshev polynomials. The Chebyshev polynomials of the first kind $T_n(x)$ are generated by

$$G(x, h) = \frac{1 - xh}{1 - 2xh + h^2} \tag{5}$$

which gives

$$\left.\begin{aligned}
T_0(x) &= 1\\
T_1(x) &= x\\
T_2(x) &= -1 + 2x^2\\
T_3(x) &= -3x + 4x^3\\
T_4(x) &= 1 - 8x^2 + 8x^4\\
T_5(x) &= 5x - 20x^3 + 16x^5\\
T_6(x) &= -1 + 18x^2 - 48x^4 + 32x^6\\
T_7(x) &= -7x + 56x^3 - 112x^5 + 64x^7\\
T_8(x) &= 1 - 32x^2 + 160x^4 - 256x^6 + 128x^8
\end{aligned}\right\} \tag{6}$$

I have carried that display far enough to expose its many patterns, which I will however not linger to discuss. I look instead to reformulation of the generating function and to the production of direct descriptions of the polynomials.

We notice that

$$\frac{1}{1 - h/\alpha} + \frac{1}{1 - h/\beta} = \frac{2\alpha\beta - (\alpha + \beta)h}{(\alpha - h)(\beta - h)} = \frac{2 - 2xh}{1 - 2xh + h^2} = 2G(x, h)$$

so we have

$$\begin{aligned}
G(x, h) &= \sum_{n=0}^{\infty} T_n(x) h^n\\
&= \tfrac{1}{2}\big\{(1 - h/\alpha)^{-1} + (1 - h/\beta)^{-1}\big\}\\
&= \tfrac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{1}{\alpha^n} + \frac{1}{\beta^n}\Big)h^n
\end{aligned} \tag{7}$$

giving

$$\begin{aligned}
T_n(x) &= \tfrac{1}{2}(\alpha^{-n} + \beta^{-n})\\
&= \tfrac{1}{2}(\alpha^{+n} + \beta^{+n}) \qquad \text{by } \alpha\beta = 1\\
&= \tfrac{1}{2}(\alpha^{+n} + \alpha^{-n})
\end{aligned} \tag{8}$$

From these pretty results it is but a quick step to Shpilkin's point of departure, for from

$$e^{i\omega} = \cos\omega + i\sqrt{1-\cos^2\omega}$$

one has the identity

$$\omega = -i\log(\cos\omega + \sqrt{\cos^2\omega - 1})$$

which (set $\cos\omega = x$) can be written

$$\omega = \arccos x = -i\log(x + \sqrt{x^2-1}) = -i\log\alpha$$

Therefore

$$\begin{aligned}\cos n\omega &= \tfrac{1}{2}(e^{in\omega} + e^{-in\omega})\\ &= \tfrac{1}{2}(e^{n\log\alpha} + e^{-n\log\alpha})\\ &= \tfrac{1}{2}(\alpha^n + \alpha^{-n})\end{aligned}$$

which by (8) gives

$$T_n(x) = \cos(n\arccos x) \tag{9}$$

It is from (9)—which, as I have mentioned, does not appear in Spanier & Oldham's *Atlas of Functions* or other sources immediately available to me—that Shpilkin extracts his elegantly swift derivation of the differential equations satisfied by the polynomials $T_n(x)$. Trivially,

$$\tfrac{d^2}{dy^2}\cos ny + n^2\cos ny = 0$$

which when $y = y(x)$ becomes

$$\big(\tfrac{dx}{dy}\tfrac{d}{dx}\big)^2\cos ny(x) + n^2\cos ny(x) = 0$$

In the case at hand $y(x) = \arccos x$ so $x(y) = \cos y$, $\frac{dx}{dy} = -\sin y = -\sqrt{1-x^2}$ and we have

$$\big(\sqrt{1-x^2}\,\tfrac{d}{dx}\big)^2 T_n(x) + n^2 T_n(x) = 0$$

which by $\big(\sqrt{1-x^2}\,\frac{d}{dx}\big)^2 = (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx}$ gives

$$(1-x^2)T_n'' - xT_n' + n^2T_n = 0 \tag{10}$$

We note in passing that the T_n-generator[2]

$$\mathcal{G}(x,h) = 1 - \tfrac{1}{2}\log(1 - 2xh + h^2)$$

listed by Abramowitz & Stegun is distinct from the $G(x,h)$ of (5). But

$$\begin{aligned}
\mathcal{G}(x,h) &= 1 - \tfrac{1}{2}\log[(1-h/\alpha)(1-h/\beta)] \\
&= 1 - \tfrac{1}{2}\log(1-h/\alpha) - \tfrac{1}{2}\log(1-h/\beta) \\
&= 1 + \tfrac{1}{2}\sum_{n=1}^{\infty}\tfrac{1}{n}(h/\alpha)^n + \tfrac{1}{2}\sum_{n=1}^{\infty}\tfrac{1}{n}(h/\beta)^n \\
&= 1 + \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{1}{\alpha^n}+\frac{1}{\beta^n}\Big)h^n \\
&= T_0(x) + \sum_{n=1}^{\infty}\tfrac{1}{n}T_n(x)\,h^n
\end{aligned}$$

so the difference is mainly cosmetic.

Legendre polynomials. The Lengendre polynomials are generated by

$$H(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} \tag{11}$$

which gives

$$\left.\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}(-1+3x^2) \\
P_3(x) &= \tfrac{1}{2}(-3x+5x^3) \\
P_4(x) &= \tfrac{1}{8}(3-30x^2+35x^4) \\
P_5(x) &= \tfrac{1}{8}(15x-70x^3+63x^5) \\
P_6(x) &= \tfrac{1}{16}(-5+105x^2-315x^4+231x^6) \\
P_7(x) &= \tfrac{1}{16}(-35x+315x^3-693x^5+429x^7)
\end{aligned}\right\} \tag{12}$$

in which again many patterns are evident, but are of no present concern. The familiar factorization provides

$$\begin{aligned}
H(x,h) &= \frac{1}{\sqrt{(1-h/\alpha)(1-h/\beta)}} \\
&= \frac{1}{\sqrt{(1-h/\alpha)(1-\alpha h)}}
\end{aligned} \tag{13}$$

We find it more efficient to work from the latter, since that avoids having to draw after the fact upon $\beta = \alpha^{-1}$. The function (13) lacks the neat structure of (7), has not the form of an explicit expansion in powers of h so does not provide ready-made descriptions of the polynomials $P_n(x)$; it is by *Mathematica*-assisted calculation that we obtain this formulation of (12):

$$\left.\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= \tfrac{1}{2\alpha}(1+\alpha^2)\\
P_2(x) &= \tfrac{1}{8\alpha^2}(3+2\alpha^2+3\alpha^4)\\
P_3(x) &= \tfrac{1}{16\alpha^3}(5+3\alpha^2+3\alpha^4+5\alpha^6)\\
P_4(x) &= \tfrac{1}{128\alpha^4}(35+20\alpha^2+18\alpha^4+20\alpha^6+35\alpha^8)\\
P_5(x) &= \tfrac{1}{256\alpha^5}(63+35\alpha^2+30\alpha^4+30\alpha^6+35\alpha^8+63\alpha^{10})
\end{aligned}\right\} \tag{12}$$

Which are not so useless as they may—with their attractive symmetry—appear. For making use again of $\omega = \arccos x = -i\log\alpha$ we have

$$\alpha = e^{i\omega} \quad \text{with} \quad \cos\omega = x$$

and with the assistance of *Mathematica*'s `ExpToTrig` command obtain

$$\left.\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= \cos\omega\\
P_2(x) &= \tfrac{1}{4}(1+3\cos 2\omega)\\
P_3(x) &= \tfrac{1}{8}(3\cos\omega+5\cos 3\omega)\\
P_4(x) &= \tfrac{1}{64}(9+20\cos 2\omega+35\cos 4\omega)\\
P_5(x) &= \tfrac{1}{128}(30\cos\omega+35\cos 3\omega+63\cos 5\omega)
\end{aligned}\right\} \tag{13}$$

which are of use in applications, particularly to the applied theory of spherical harmonics.

The results developed above provide no formula comparable to (9), so provide no basis on which to construct a similarly swift derivation of Legendre's differential equation

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

In this respect they demonstrate a respect in which the Chebyshev polynomials are special, and the utility of my general recursive method.[2]

Comparison with Mayer's method. Ahmed Sebbar's polynomials (of what I call the 1st kind[3]) $S_n(x)$ arise in the usual way from a generating function

$$F(x,h) = \log(1-3xh-h^3) \tag{14}$$

into which h enters cubically. The polynomials have been demonstrated[4] to satisfy linear differential equations of third order

$$(4x^3+1)S_n''' + 18x^2S_n'' - (3n^2+3n-10)xS_n' - n^2(n+3)S_n = 0 \tag{15}$$

[3] Sebbar has interest in four distinct populations of polynomials, with closely related cubic generators.

[4] See "Ray Mayer's reconstruction of Ahmed Sebbar's DE" (a *Mathematica* notebook in PDF format, November 2017) and "Extracting differential equations. . . ,"[2] pages 9–11.

Mayer's approach to the derivation of that equation makes essential use of a generalized factorization scheme. He writes

$$1 - 3xh - h^3 = (\alpha - h)(\beta - h)(\gamma - h) \tag{16}$$

Comparison of the expression on the left with the expansion of the exprssion on the right gives

$$\left.\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha\beta + \beta\gamma + \gamma\alpha &= 3x \\ \alpha\beta\gamma &= 1 \end{aligned}\right\} \tag{17}$$

where $\{\alpha, \beta, \gamma\}$, the roots of $1 - 3xh - h^3 = 0$ (unique to within permutations), are solutions of (17). Those, as reported by *Mathematica*, are enormously complex, but can with patience be shown to have the structure

$$\left.\begin{aligned} \alpha &= \omega_1 A^{-1} + \omega_1^2 Ax \\ \beta &= \omega_2 A^{-1} + \omega_2^2 Ax \\ \gamma &= \omega_3 A^{-1} + \omega_3^2 Ax \end{aligned}\right\} \tag{18.1}$$

where $\{\omega_1, \omega_2, \omega_3\}$ are cube roots of -1:

$$\begin{aligned} \omega_1 &= -1 \\ \omega_2 &= e^{+i\pi/3} = \tfrac{1}{2}(1 + i\sqrt{3}) \\ \omega_2 &= e^{-i\pi/3} = \tfrac{1}{2}(1 - i\sqrt{3}) \end{aligned}$$

Equations (18.1) give

$$\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha\beta + \beta\gamma + \gamma\alpha &= 3x \\ \alpha\beta\gamma &= (A^6x^3 - 1)/A^3 \end{aligned}$$

so to achieve (17) the function $A(x)$ must be a solution of $(A^6x^3 - 1)/A^3 = 1$, which is to say: $A = a^{1/3}$, where

$$a(x) = \frac{1 \pm \sqrt{1 + 4x^3}}{2x^3}$$

is a solution of $(a^2x^3 - 1)a = 1$. Evidently there are six such A-functions, and it is a matter of indifference which one we adopt; Ray Mayer elected[4] to work with

$$A = \frac{2^{1/3}}{\left(-1 + \sqrt{1 + 4x^3}\,\right)^{1/3}} \tag{18.2}$$

From (16) we by $\alpha\beta\gamma = 1$ have

$$1 - 3xh - h^3 = (1 - h/\alpha)(1 - h/\beta)(1 - h/\gamma)$$

Returning with this result to (14), we have

$$
\begin{aligned}
F(x,h) &= \log(1-h/\alpha)+\log(1-h/\beta)+\log(1-h/\gamma)\\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{1}{\alpha^n}+\frac{1}{\beta^n}+\frac{1}{\gamma^n}\Big)h^n\\
&= \sum_{n=1}^{\infty}S_n(x)h^n \equiv -\sum_{n=1}^{\infty}\frac{1}{n}M_n(x)h^n
\end{aligned}
\tag{19}
$$

This construction of $S_n(x)$ bears a striking resemblance to (7), and more particularly to a formula that appears on page 4. We expect such formulæ to arise whenever

$$
\begin{aligned}
\text{generating function} &= \log(1+\cdots\pm h^p)\\
&= \log[(\alpha_1-h)(\alpha_2-h)\cdots(\alpha_p-h)]
\end{aligned}
$$

where $\cdots$ signifies a polynomial of the form $\sum_{k=1}^{p-1}c_k(x)h^k$.

It was from (19) that Mayer worked to obtain the differential equations satisfied by the polynomials $M_n(x)$, whence by $S_n(x)$. His argument was made computationally heavy by the complexity (18) of the functions $\{\alpha,\beta,\gamma\}$. It was the relative simplicity (and, indeed, the special structure (3)) of the functions $\{\alpha,\beta\}$ that made it possible to proceed from (8) by a natural change of variable to the elegant formula (9), from which the associated differential equations followed swiftly. The question arises: Can one in the present context construct an analog of (9) from which Sebbar's DE can be obtained with similar swiftness?

I do not know the answer, am presently inclined to think it is "no." But record one possibly relevant thought. The argument presented on page 3 makes essential use of a property

$$
\tfrac{d^2}{dx^2}\cos nx = -n^2\cos nx
$$

of the real-valued function

$$
\cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{e^{(-1)^{1/2}x}+e^{(-1)^{-1/2}x}}{2}
$$

and its inverse. One might therefore expect in the present context to have need of the real-valued function

$$
f(x) = \frac{e^{(-1)^{1/3}x}+e^{(-1)^{-1/3}x}}{2} = e^{x/2}\cos(\tfrac{1}{2}\sqrt{3}\,x)
$$

which satisfies

$$
\tfrac{d^3}{dx^3}f(nx) = -n^3 f(nx)
$$

As so also do

$$
\begin{aligned}
g(nx) &= e^{nx/2}\sin(\tfrac{1}{2}\sqrt{3}\,nx)\\
h(nx) &= e^{-nx}
\end{aligned}
$$

The function $h(nx)$ is monotonic, so possesses a functional inverse; indeed,

$$y = h(nx) \quad\Longrightarrow\quad x = \log(1/y^n)$$

But the growth of $f(nx)$ and $g(nx)$ is oscillatory (= not monotonic) so their functional inverses are multivalued, so defined only locally. This train of thought appears to take us no closer to the construction of a Sebbar analog of (9).

Construction of polynomials that satisfy high-order differential equations. The classic orthogonal polynomials all satisfy linear differential equations of second order. It is in this light somewhat surprising that Sebbar's polynomials $S_n(x)$ satisfy equations of *third* order. Alexander Shpilkin was led therefore to wonder[1] whether one can construct polynomials that satisfy DEs of arbitrarily great order.[5] I approach the question by explaining how it comes about that Sebbar's polynomials satisfy DEs of 3$^{\rm rd}$ order. I will allow myself to report as bald assertions facts that were obtained by *Mathematica*-based experimentation.

Mathematica reports that the general solution of Sebbar's DE (15) is a linear combination of the following functions:

$$\left.\begin{aligned}
H_0(n) &= x^0\cdot {}_3F_2\big(\big\{\tfrac12+\tfrac{n}{6},-\tfrac{n}{3},\tfrac{n}{6}\big\},\big\{\tfrac13,\tfrac23\big\},-4x^3\big)\\
H_1(n) &= x^1\cdot {}_3F_2\big(\big\{\tfrac13-\tfrac{n}{3},\tfrac13+\tfrac{n}{6},\tfrac56+\tfrac{n}{6}\big\},\big\{\tfrac23,\tfrac43\big\},-4x^3\big)\\
H_2(n) &= x^2\cdot {}_3F_2\big(\big\{\tfrac23-\tfrac{n}{3},\tfrac23+\tfrac{n}{6},\tfrac76+\tfrac{n}{6}\big\},\big\{\tfrac43,\tfrac53\big\},-4x^3\big)
\end{aligned}\right\}\tag{20}$$

where the generalized hypergeometric functions are defined

$${}_3F_2\big(\big\{a_1,a_2,a_3\big\},\big\{b_1,b_2\big\},y\big) = \sum_{k=0}^\infty \frac{1}{k!}\frac{[a_1,k][a_2,k][a_3,k]}{[b_1,k][b_2,k]}\,y^k \tag{21}$$

and where the expressions $[\bullet,\bullet]$ are Pochhammer symbols (ascending factorials), the meaning of which[6] is illustrated by

$$[a,5] = a(1+a)(2+a)(3+a)(4+a)$$

Clearly

$$\left.\begin{aligned}
[0,k] &= 0 \quad:\quad k\geqslant 1\\
[-1,k] &= 0 \quad:\quad k\geqslant 2\\
&\ \ \vdots\\
[-n,k] &= 0 \quad:\quad k\geqslant n+1 \quad (n\text{ a positive integer})
\end{aligned}\right\}\tag{22}$$

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[5] If $P(x)$ satisfies a DE of order n then it satisfies also the DEs of order $m>n$ that are produced by m-fold differentiation of the original DE. Our concern here, therefore, is with DEs of arbitrarily great *least/minimal order* n.

[6] See Spanier & Oldham, Chapter 18. $[a,0]=1$ by definition.

to truncate; *i.e.*, to reduce to a polynomial.

I show how this works in the production of Sebbar polynomials. Look to the parameter sets that appear in (20):

$$\begin{aligned}
h_0(n) &= \left[\left\{\tfrac{1}{2}+\tfrac{n}{6}, -\tfrac{n}{3}, \tfrac{n}{6}\right\}, \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\}\right]\\
h_1(n) &= \left[\left\{\tfrac{1}{3}-\tfrac{n}{3}, \tfrac{1}{3}+\tfrac{n}{6}, \tfrac{5}{6}+\tfrac{n}{6}\right\}, \left\{\tfrac{2}{3}, \tfrac{4}{3}\right\}\right]\\
h_2(n) &= \left[\left\{\tfrac{2}{3}-\tfrac{n}{3}, \tfrac{2}{3}+\tfrac{n}{6}, \tfrac{7}{6}+\tfrac{n}{6}\right\}, \left\{\tfrac{4}{3}, \tfrac{5}{3}\right\}\right]
\end{aligned}$$

We find

$$\begin{aligned}
h_0(0) &= \big[\{\bullet, 0, 0\}, \{\bullet, \bullet\}\big]\\
h_1(0) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_2(0) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]
\end{aligned}$$

$$\begin{aligned}
h_0(1) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_1(1) &= \big[\{0, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_2(1) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]
\end{aligned}$$

$$\begin{aligned}
h_0(2) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_1(2) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_2(2) &= \big[\{0, \bullet, \bullet\}, \{\bullet, \bullet\}\big]
\end{aligned}$$

$$\begin{aligned}
h_0(3) &= \big[\{\bullet, -1, \bullet\}, \{\bullet, \bullet\}\big]\\
h_1(3) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_2(3) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]
\end{aligned}$$

$$\begin{aligned}
h_0(4) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_1(4) &= \big[\{-1, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_2(4) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]
\end{aligned}$$

$$\begin{aligned}
h_0(5) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_1(5) &= \big[\{\bullet, \bullet, \bullet\}, \{\bullet, \bullet\}\big]\\
h_2(5) &= \big[\{-1, \bullet, \bullet\}, \{\bullet, \bullet\}\big]
\end{aligned}$$

The pattern continues, with -2 replacing -1, etc. We conclude that

$$\begin{matrix}
H_0(0) & H_0(3) & H_0(6) & \dots\\
H_1(1) & H_1(4) & H_1(7) & \dots\\
H_2(2) & H_2(5) & H_2(8) & \dots
\end{matrix}$$

are polynomials, and find that they are (to within numerical factors) precisely the Sebbar polynomials

$$\begin{matrix}
S_0(x) & S_3(x) & S_6(x) & \dots\\
S_1(x) & S_4(x) & S_7(x) & \dots\\
S_2(x) & S_5(x) & S_8(x) & \dots
\end{matrix}$$

Generalized hypergeometric functions of the form (21) are known to satisfy third-order linear differential equations of the form

$$y(Y+a_1)(Y+a_2)(Y+a_3)f(y) = Y(Y+b_1-1)(Y+b_2-1)f(y)$$

where the differential operator

$$Y = y\tfrac{d}{dy} = \tfrac{d}{dy}y - 1$$

and where the operators $(Y+a_1)$ and $(Y+a_2)$ are seen to commute:

$$(Y+a_1)(Y+a_2) = (Y+a_2)(Y+a_1) = Y^2 + (a_1+a_2)Y + a_1a_2$$

Simple change-of-variable techniques permit one to construct the DE satisfied by ${}_3F_2\big(\{a_1,a_2,a_3\},\{b_1,b_2\},g(x)\big)$, and thus to recover Sebbar's DE (15). The extension to functions of the form ${}_pF_q\big(\{a_1,a_2,\ldots,a_p\},\{b_1,b_2,\ldots b_q\},g(x)\big)$, which we expect to satisfy DEs of a minimal order which is the greater of $\{p,q+1\}$, is straightforward.

The preceding discussion indicates how one might proceed to construct polynomials that satisfy differential equations of arbitrarily great minimal order. One might expect to play similar games with Meijer G-functions and Fox H-functions, generalizations of the hypergeometric functions.